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Improved Warning Limits Control Chart to Detect changes in the Mean Vector

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Abstract

Rattihalli et al. (2021-2022) proposed a control chart with warning limits. Also, Gadre and Rattihalli (2022) proposed a control chart by considering a head start without warning limits. For univariate processes, Gadre and Patel (2024) proposed a control chart called 'Improved Warning Limits control chart to detect shifts in the process mean' (IWL-$\overline{X}$) chart by taking the combination of these two and changing a head start. For a multivariate case, a control chart called 'Improved Warning Limits control chart to monitor changes in the mean vector' (IWL-χ^2) chart is proposed. This chart is an extended version of IWL-$\overline{X}$ chart. From the numerical study, it is verified that, IWL-χ^2 chart significantly reduces the out of control 'Average Time to Signal' (ATS)as compared to the χ^2chart, 'Modified Control Chart with Warning Limits' ($MCCWL$-χ^2) and 'Improved Control Chart' (ICC-χ^2).

Key Words: Control Limits, Warning Limits, Average Run Length (ARL), Average Time to Signal (ATS), Modified Control Chart with Warning Limits ($MCCWL$-χ^2) and Improved Control Chart (ICC-χ^2) to monitor mean vector.

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1. Introduction

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For multivariate processes, Hotelling (1947) developed T^2 and χ^2 charts. Mahadik (2012) proposed 'Variable Control and Warning Limits to monitor the process mean vector' (*VCWL*-T^2) chart by using sampling interval as well as the warning limits. Lowry et al. (1992) proposed a 'Multivariate *EWMA*' (*MEWMA*) control chart. Prabhu and Runger (1997) carried out a detailed analysis of the *ARL* performance of the *MEWMA* control chart. Crosier (1988) and Pignatiello and Runger (1990) have explained several 'Multivariate CUSUM' (*MCUSUM*) procedures.

For χ^2 chart and T^2 chart, the production process is considered as in control, if the current value of the charting statistic ($\chi^2$and T^2) is within the control limits, irrespective of its departure from the central line. A sample point being close to the control limits, though within the control limits, gives stronger evidence to suspect the quality of the production process. Thus, it is desirable to consider the relative position of a point in assessing the quality of the production process. This aspect is accounted in control charts with the warning limits. Li et al. (2014) proposed various methods for computations of *ARL* and *ATS*. Some of them are Markov chain approach and integration equation method. Wu et al. (2001) introduced the term *ATS*.

Rattihalli et al. (2021-2022) introduced 'Modified Control Chart with Warning Limits to detect shifts in the mean vector' (*MCCWL*-χ^2) to monitor the mean vector of a production process. They used the Markov chain approach to derive *ARL* and hence the *ATS* expressions of the *MCCWL*-χ^2. In *MCCWL*-χ^2, it assumed that, at time zero, $\chi^2 \in A$ (Acceptance Region). Also, Gadre and Rattihalli (2022) developed 'Improved Control Chart to monitor the mean vector' (*ICC*-χ^2).

It is to be noted that, though in *MCCWL*-χ^2, it is assumed that, at time zero, $\chi^2 \in A$, χ^2 may be much closer to the control limit. In such a case, assume that, at time zero, $\chi^2 \in W$ (Warning Region). By taking into consideration of this fact, a control chart called the 'Improved

Warning Limits control chart to monitor the mean vector' (*IWL*-χ^2) chart is proposed. Let '*R*' be the rejection region of *IWL*-χ^2 chart. Under the assumption mentioned above, the stopping rule of *IWL*-χ^2 control chart is 'Declare the process as out of control if $\chi^2 \in R$ or two successive sample points $\chi^2 \in W$'. For multivariate processes, *ATS* performance of the proposed chart is better as compared to the χ^2 chart, *MCCWL*-χ^2 and *ICC*-χ^2.

Section wise organization of the remainder of this article is as below. In **Section 2**, a brief review of the control charts is discussed. **Section 3** covers the basic notations, the operation and the derivation of *ATS* expression of the proposed chart. Some numerical illustrations and comparative study of *IWL*-χ^2chartwith the χ^2 chart, *MCCWL*-χ^2 and *ICC*-χ^2 is carried out in **Section 4**. Concluding remarks are given in **Section 5**.

2. Brief Review of the control charts

Hotelling H. (1947) introduced the χ^2 chart and T^2 control chart which are used to monitor the mean vector and the operation is based only on the most recent observation, Therefore it is insensitive to detect small and moderate shifts in the mean vector. Let $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n$ be a random sample from $N_p(\underline{\mu}, \boldsymbol{\Sigma})$ distribution. Here, $\underline{\mu}$ is the process mean vector and $\boldsymbol{\Sigma}$ is the known covariance matrix.

2.1. Hotelling's T^2 chart with variable control and warning limits

Mahadik (2012) developed a 'Hotelling's T^2 chart with variable control and warning limits' (*VCWL*-T^2). Numerical comparison of the T^2 chart and *VCWL*-T^2 chart is carried out and it is observed that the *ARL* performance of the *VCWL*-T^2 is efficient as compared to that of the T^2 chart. As the in control mean vector is known, it is better to treat *VCWL-* χ^2 chart instead of *VCWL-*T^2 chart.

2.2. Modified Control Chart with Warning Limits to detect shifts in the mean vector (*MCCWL-*χ^2):

Rattihalli et al. (2021-2022) developed *MCCWL-*χ^2. In *MCCWL-*χ^2, let A, W and R be the acceptance, warning and the rejection regions of the *MCCWL--*χ^2 respectively. Corresponding to the observed sample of size n_{mwch} (, Say), let χ^2 be the charting statistic. In *MCCWL-*χ^2, it is assumed that χ^2_{oz} (the charting statistic χ^2 at time zero) is in the acceptance region. Under the assumption of *MCCWL-*χ^2 mentioned above, for r = 1, 2, …, the stopping rule is 'Declare the process as out of control if $\chi^2_r \in R$ or $\chi^2_r \in W$ and $\chi^2_{(r+1)} \in W$. Let $P_A = P\ (\chi^2 \epsilon\ A)$. Also, the terms P_W and P_R have similar meaning. Under the assumption that, $\chi^2_{oz} \in A$, *ATS* expression of *MCCWL-*χ^2 is $\frac{n_{mwch(1+P_W)}}{1-P_A(1+P_W)}$. *MCCWL-*$\chi^2$ performs better as compared to the χ^2 chart.

2.3. Improved Control Chart to detect shift in the mean vector (*ICC-*χ^2):

Gadre and Rattihalli (2022) developed *ICC-*χ^2. In *ICC-*χ^2, let 'A' and 'R' be the acceptance and rejection regions of the (***ICC-***χ^2) respectively. Here, it is assumed that $\chi^2_{oz} \in R$. The stopping rule of this chart is, 'Stop if the charting statistic χ^2 of the first sample (χ^2_1, Say) falls in R; otherwise for r> 1, if $\chi^2_r \in R$ and $\chi^2_{(r+1)} \in R$. If it is assumed that $\chi^2_{oz} \in R$, this rule can be written as for r (r = 0,1, …), 'Declare the process as out of control if two successive χ^2 fall outside the control limits'. With the head start of this chart, the *ATS* expression of *ICC-*χ^2 is $\frac{n_{ich}}{P_R^2}$. It is numerically illustrated that, *ICC*-χ^2 performs significantly better as compared to the *MCCWL*-χ^2 and the χ^2 chart.

In the next section, the notations, operation and *ATS* expression of the *IWL*-χ^2 chart is derived.

3. Basic Notations, the Operation, *ATS* criterion and Derivation of the *ATS* expression

This section covers the notations, stepwise procedure and the *ATS* expression of *IWL*-χ^2 chart. Let p be the number of quality characteristics. For simplicity take $p = 2$ and let X, Y be the quality characteristics.

3.1. *Basic Notations:*

The following are the notations related to *IWL*-χ^2 chart.

1. n_{iwch}: The sample size.
2. $\underline{\mu} = (\mu_x, \mu_y)$: The process mean vector.
3. $\underline{\mu}_0 = (\mu_{0x}, \mu_{0y})$: In control values of the mean vector.
4. $\underline{\mu}_1 = (\mu_{1x}, \mu_{1y})$: Out of control values of the mean vector.
5. (σ_x, σ_y): The process variability of the respective quality characteristics.
6. $\boldsymbol{\Sigma_0}$: The known covariance matrix of *IWL*-χ^2 chart, which depends on the correlation coefficientρ.
7. $\underline{\delta}_1 = (\delta_{1x}, \delta_{1y}) = \left(\frac{\mu_{1x} - \mu_{0x}}{\sigma_{\bar{y}}}, \frac{\mu_{1y} - \mu_{0y}}{\sigma_{\bar{y}}}\right)$, where $\sigma_{\bar{X}} = \frac{\sigma_x}{\sqrt{n}}, \sigma_{\bar{Y}} = \frac{\sigma_y}{\sqrt{n}}$ for a bivariate process and it is shift in the standardized mean vector.
8. $\underline{\bar{X}}$: Sample mean vector.

9. $\chi^2 = n_{iwch}\left(\underline{\overline{X}} - \underline{\mu}\right)^{/} \Sigma_0 \left(\underline{\overline{X}} - \underline{\mu}\right)$: The test statistic of the *IWL*-χ^2 chart.

10. $\chi_0^2 = n_{iwch}\left(\underline{\overline{X}} - \underline{\mu}_0\right)^{/} \Sigma_0^{-1} \left(\underline{\overline{X}} - \underline{\mu}_0\right)$: The test statistic for in control *IWL*-χ^2 chart.

11. ρ: The correlation coefficient between the two quality characteristics X and Y.

12. k_{iwch}: Upper control limit of the *IWL*-χ^2 chart.

13. k_{1iwch}: Upper warning limit of the *IWL*-χ^2 chart.

14. $A = (0, k_{1iwch})$, $R = (k_{iwch,} \infty)$ and $W = (k_{1iwch,} k_{iwch,})$ are the acceptance, rejection and warning regions respectively.

15. $p_{0A} = p\left(\chi^2 < k_{1iwch} \mid \underline{\mu} = \underline{\mu}_0\right)$, $p_{0w} = p\left(k_{1iwch} < \chi^2 < k_{iwch} \mid \underline{\mu} = \underline{\mu}_0\right)$ and $p_{0R} = 1 - p_{0A} - p_{0w}$ are the respective in control probabilities.

16. $p_{1A} = p\left(\chi^2 \leq k_{1iwch} \mid \underline{\mu} = \underline{\mu}_1\right)$, $p_{1w} = p\left(k_{1iwch} \ll \chi^2 \leq k_{iwch} \mid \underline{\mu} = \underline{\mu}_1\right)$ and $p_{1R} = 1 - p_{1A} - p_{1w}$ are the respective out of control probabilities.

17. $ARL(\underline{\delta})$: Average number of samples inspected by the time the process has gone out of control to monitor changes in the mean vector from $\underline{\mu}_0$ to $\underline{\mu}$.

18. ARL_0 and ARL_1: In-control and out-of-control *ARLs*.

19. $ATS(\underline{\delta})$: Average number of units inspected by the time the process has gone out of control to detect shift in the mean vector from $\underline{\mu}_0$ to $\underline{\mu}$. Note that, $ATS(\underline{\delta})$: $n_{iwch}(ARL(\underline{\delta}))$. Also, ATS_0andATS_1: In-control and out-of-control *ATSs*.

20. τ: The minimum required value of ATS_0, the average time to signal when the process is in control.

3.2.*Operation and the derivation of ATS expression of the IWL-χ^2 chart:*

Let *CNT* be the counter. A stepwise procedure of the implementation of *IWL-*χ^2chartis as below.

Step-I: Initialize *CNT* to unity and *i* to zero.

Step-II: $i = i + 1$. Take i^{th} sample of size n_{iwch} and compute χ_i^2.

Step-III: If $\chi_i^2 \; \epsilon \; A$, take *CNT* as 0 and move back to **Step-II**, else go to the next step.

Step-IV: If $\chi_i^2 \; \epsilon \; R$, move to **Step-6**, else add *CNT* by unity and then go to the next step.

Step-V: If *CNT*< 2, go back to **Step-II**, else move to the next step.

Step-VI: The process has gone out of control. In such a situation, find the assignable causes. Take the corrective action and go back to **Step-1**.

Design parameters of the *IWL-*χ^2 chart are determined using the *ATS* model. The *ATS* model is,

$$\left.\begin{array}{c} \text{Minimize } ATS_1 \\ \text{Subjectto the condition} \\ ATS_0 \geq \tau \end{array}\right\} \qquad (1)$$

To find the *ATS* expression of *IWL-*χ^2chart, *ARL* expression is derived first. As mentioned in **Sub-Section (*3.1*),** *A*, *W* and *R* are the acceptance, warning and rejection regions and '*S*' (Signal) is the absorbing state respectively.

When the production process is of 'independent and identically distributed' (i.i.d.) random variables and the sample size n_{iwch} is fixed, the sequence $\{\chi_i^2\}_{i=1}^{\infty}$of the number in defectives in i^{th} sample are also i.i.d. random variables. For *IWL-*χ^2 chart, it is enough to absorbing state is S. In AA, the first 'A' indicates a statistic χ^2 of the sample just before the current sample, was in the acceptance region; and the second 'A' indicates that of the current sample is also in the acceptance region. Similar interpretations are for the states AW and WA. The absorbing state is $S = \{AR \text{ U } WR \text{ U } WW\}$. With the state space $\{AA, AW, WA, S\}$ the expression for ARL of the $IWL\text{-}\chi^2$ chart can be obtained. The 'transition probability matrix' (t.p.m.) $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \begin{array}{c} \\ AA \\ AW \\ WA \\ S \end{array} \begin{array}{c} \begin{array}{cccc} AA & AW & WA & S \end{array} \\ \begin{pmatrix} P_A & P_W & 0 & P_R \\ 0 & 0 & P_A & P_W + P_R \\ P_A & P_W & 0 & P_R \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad (2)$$

In Equation (2) above, $P_A = \text{P}(\chi^2 \leq k_{1iwch})$, $P_R = \text{P}(\chi^2 > k_{iwch})$ and $P_W = \text{P}(\chi^2 \leq c_{iwch})$ - $\text{P}(\chi^2 \leq k_{1iwch})$. Note that, $S = (AR \text{ U } WR \text{ U } WW)$.

Let $\boldsymbol{P_1}$ be a matrix obtained by deleting the row and column of $\boldsymbol{P}$ corresponding to the state S. $\underline{ARL}$ is $(\boldsymbol{I}\text{-}\boldsymbol{P_1})^{-1}\underline{\boldsymbol{1}}$, where $\underline{\boldsymbol{1}}$ is a column vector of unit elements. Now,

$$\begin{aligned} |\boldsymbol{I}\text{-}\boldsymbol{P_1}| &= (1-P_A)(1-P_AP_W) + P_W(-P_A)^2 \\ &= 1-P_A - P_AP_W \\ &= 1-P_A(1+P_W). \end{aligned} \qquad (3)$$

Also,

$$\textbf{Adj. (I-P}_I\textbf{)} = \begin{array}{c} \\ AA \\ AW \\ WA \end{array} \begin{array}{c} \begin{array}{ccc} AA & AW & WA \end{array} \\ \left(\begin{array}{ccc} 1-P_A P_W & P_W & P_A P_W \\ (P_A)^2 & 1-P_A & P_A(1-P_A) \\ P_A & P_W & 1-P_A \end{array} \right) \end{array} \qquad (4)$$

As mentioned in Brook and Evans (1972), $\underline{ARL(\delta)} = \mathbf{(I-P_I)^{-1}\underline{1}}$,

$$\underline{ARL(\delta)} = \begin{bmatrix} \frac{1+P_W}{1-P_A\langle 1+p_W\rangle} \\ \frac{1}{1-P_A\langle 1+p_W\rangle} \\ \frac{1+P_W(p)}{1-P_A\langle 1+p_W\rangle} \end{bmatrix} \qquad (5)$$

Under the assumption about the head start of $IWL-\chi^2$ chart, from Brook and Evans (1972) and as $ATS = n_{iwch}(ARL)$, ATS of this chart is

$$ATS = \frac{n_{iwch}}{1-P_A\langle 1+P_W\rangle}. \qquad (6)$$

In the following, numerical illustrations are considered to compare the performance of $IWL-\chi^2$ chart with the χ^2 chart, $MCCWL-\chi^2$ and $ICC-\chi^2$. A MAT-Lab code is developed to obtain the design parameters together withthe ATS_I values of the four charts. Next, for numerical illustrations considered, ATS performance of the IWL-χ^2 chart with the ACS chart, χ^2 chart, $MCCWL$-χ^2 and ICC-χ^2 is carried out.

4. Numerical Illustrations

Example1:

Let $\underline{\delta}_1$ = (0, 0.75), ρ = 0.7 and τ = 370. For these input parameters, values of the design parameters of the χ^2 chart, *MCCWL-*χ^2, *ICC-*$\chi^2$and *IWL-*χ^2chart along with respective ATS_1 values are given below.

(i) χ^2chart: n_{ch}= 8, k_{ch} = 7.67, ATS_1 = 12.3516

(ii) *MCCWL-*χ^2:n_{mch}=8, k_{1mch} = 5.01, k_{mch} = 8.06, ATS_1 = 12.2129

(iii) *ICC-*χ^2:n_{ich}= 7, k_{ich} = 3.97, ATS_1 = 9.8603

(iv) *IWL-*χ^2 chart: n_{iwch}= 7, k_{1iwch} = 4.01, k_{iwch} = 14.01, ATS_1 = 9.4702

Result 1:This example shows that, not only ATS_1 of the *IWL-*χ^2 chart for the process mean vector is less than ATS_1 of the related three charts, but also the sample size n_{iwch} is not exceeding that of the related three charts, which is an indication of reduction of the delay in detection of the shift.

Further, to study the behavior of *IWL-*χ^2 chart with χ^2 chart, *MCCWL-*χ^2 and *ICC-*χ^2 corresponding to the changes in $\underline{\delta}$ value, we have computed $ATS(\underline{\delta})$ values for δ_x = 0 and for various values of δ_y. Normalized $ATS(\underline{\delta})$ values of four related charts are given in Table1. Figure1 given below is a graph of normalized $ATSs$ against δ_y values.

Table1: ***ATSs*** **and Normalized** ***ATSs*** **of** χ^2 **chart,** ***MCCWL***-χ^2, ***ICC***-χ^2 **and** ***IWL***-χ^2 **chart** ***when*** $\delta_x = 0$ **for various values of** δ_y

δ_x	δ_y	*ATS*				*Normalized ATS*			
		χ^2*chart*	*MCCWL*-χ^2	*ICC*-χ^2	*IWL*-χ^2 *chart*	χ^2*chart*	*MCCWL*-χ^2	*ICC*-χ^2	*IWL*-χ^2 *chart*
0	0.1	282.3875	278.2856	287.3236	284.1366	1	0.985474	1.01748	1.006194
0	0.2	156.3721	149.4268	155.9337	151.8872	1	0.955585	0.997196	0.971319
0	0.3	82.4881	76.7462	76.5390	83.6985	1	0.930391	0.927879	1.014674
0	0.4	46.0817	42.4904	39.5630	37.8141	1	0.922067	0.85854	0.820588
0	0.5	28.1274	26.2113	22.8566	21.7643	1	0.931878	0.81261	0.773776
0	0.6	18.8930	18.0182	14.9696	14.2574	1	0.953697	0.792336	0.754639
0	0.7	13.9272	13.6241	11.0395	10.5624	1	0.978237	0.792658	0.758401
0	0.8	11.1724	11.1405	9.0087	8.6922	1	0.997145	0.806335	0.778007
0	0.9	9.6311	9.6943	7.9538	7.7550	1	1.006562	0.825845	0.805204
0	1	8.7865	8.8568	7.4219	7.3090	1	1.008001	0.844694	0.831844
0	1.1	8.3475	8.3950	7.1704	7.1144	1	1.00569	0.858988	0.852279
0	1.2	8.1379	8.1625	7.0619	7.0382	1	1.003023	0.867779	0.864867
0	1.3	8.0484	8.0588	7.0200	7.0115	1	1.001292	0.872223	0.871167
0	1.4	8.0149	8.0186	7.0057	7.0031	1	1.000462	0.874085	0.87376
0	1.5	8.0040	8.0051	7.0014	7.0008	1	1.000137	0.874738	0.874663

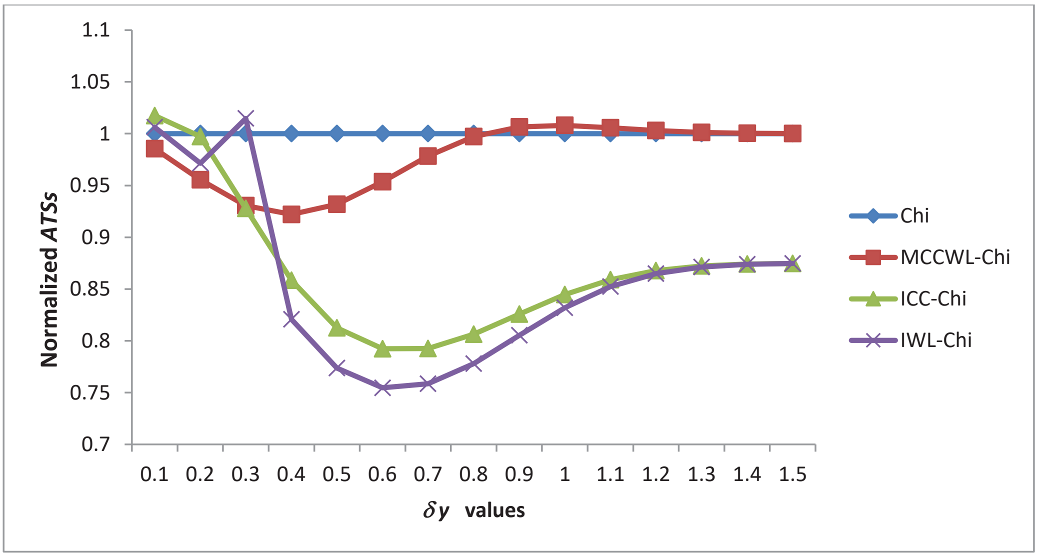

Figure1: Graph of Normalized ***ATSs*** **of the four charts**

Result 2: From Figure 1, it is observed that, for $\delta_y > 0.3$, $IWL\text{-}\chi^2$ chart signals faster than χ^2 chart, $MCCWL\text{-}\chi^2$ and $ICC\text{-}\chi^2$. When δ_y is not far away from δ_{1y}, the chart is the best, if the *ATS* of that chart is the smallest among the related three charts. Hence, $IWL\text{-}\chi^2$ chart is efficient as compared to the related three charts.

Example2:

Here, for $\rho = 0.7$, $\tau = 370$ and 14 combinations of $\underline{\delta}_1$, the design parameters of χ^2 chart, $MCCWL\text{-}\chi^2$, $ICC\text{-}\chi^2$ and $IWL\text{-}\chi^2$ chart are obtained along with ATS_1 values of each of the combinations and are given in Table 2. Also, for $\rho = 0.5$, $\tau = 370$, the design parameters and the input parameters along with the ATS_1 values of the related four charts are given in Table 3.

Table2: Design parameters of the χ^2chart, $MCCWL\text{-}\chi^2$, $ICC\text{-}\chi^2$ and $IWL\text{-}\chi^2$chart with respective ATS_1 values

$\underline{\delta}_1 = (\delta_{1x}, \delta_{1y})$	χ^2 Chart			$MCCWL\text{-}\chi^2$				$ICC\text{-}\chi^2$			$IWL\text{-}\chi^2$ Chart			
	$\rho = 0.7$			$\rho = 0.7$				$\rho = 0.7$			$\rho = 0.7$			
	n	*K*	ATS_1	*n*	k_1	*k*	ATS_1	*n*	*k*	ATS_1	*n*	k_1	*k*	ATS_1
(0, 0.50)	15	6.42	**23.2314**	14	5.01	6.66	**23.0071**	12	3.43	**19.0532**	11	4.01	8.63	**18.1887**
(0, 0.75)	8	7.67	**12.3516**	8	5.01	8.06	**12.2129**	7	3.97	**9.8603**	7	4.01	14.01	**9.4702**
(0, 1)	5	6.61	**7.7828**	5	6.01	8.82	**7.6284**	4	4.53	**6.0764**	4	5.01	10.81	**5.8039**
(0, 1.50)	3	9.63	**3.9810**	3	7.01	9.76	**3.9395**	2	5.23	**3.0438**	2	6.01	11.55	**2.9716**
(0.50, 0.50)	21	5.74	**33.9704**	20	4.01	6.12	**33.7879**	18	3.03	**28.5184**	19	3.01	10.2	**27.2821**
(0.50, 0.75)	13	6.70	**20.8782**	13	5.01	6.84	**20.6684**	11	3.52	**17.0359**	10	4.01	9.16	**16.1572**
(0.50, 1)	8	7.67	**12.5848**	8	5.01	8.06	**12.4229**	7	3.97	**10.0321**	7	4.01	14.01	**9.6285**
(0.50, 1.50)	4	9.06	**5.7212**	4	6.01	9.37	**5.6416**	3	4.82	**4.4269**	3	5.01	12.92	**4.1939**
(0.75, 0.75)	12	6.86	**18.4397**	11	5.01	7.24	**18.2421**	10	3.62	**14.9806**	9	4.01	9.89	**14.1240**
(0.75, 1)	9	7.44	**13.3028**	8	5.01	8.06	**13.1322**	7	3.07	**10.6222**	7	4.01	14.01	**10.1745**
(0.75, 1.50)	5	8.61	**6.5748**	4	6.01	9.37	**6.4010**	4	4.53	**5.1406**	3	5.01	12.92	**4.8521**
(1, 1)	8	7.67	**11.7290**	7	4.01	8.44	**11.6037**	6	4.13	**9.3409**	7	4.01	14.01	**9.0544**
(1, 1.50)	5	8.61	**6.9357**	4	6.01	9.37	**6.8288**	4	4.53	**5.4161**	4	5.01	10.81	**5.2073**
(1.50, 1.50)	4	9.06	**6.0804**	4	6.01	9.37	**5.9581**	3	4.82	**4.7287**	3	5.01	12.92	**4.4635**

Table 3: Design parameters of the χ^2 chart, *MCCWL*-χ^2, *ICC*-χ^2 and *IWL*-χ^2 chart with respective ATS_1 values

$\underline{\delta}_1 = (\delta_{1x}, \delta_{1y})$	χ^2 *Chart*			*MCCWL*-χ^2				*ICC*-χ^2			*IWL*-χ^2 *Chart*			
	$\rho = 0.5$			$\rho = 0.5$				$\rho = 0.5$			$\rho = 0.5$			
	n	*K*	ATS_1	*N*	k_1	*k*	ATS_1	*n*	*k*	ATS_1	*n*	k_1	*k*	ATS_1
(0, 0.50)	19	5.94	**30.9906**	19	4.01	6.26	**30.8542**	16	3.15	**25.8896**	19	3.01	11.87	**25.1914**
(0, 0.75)	11	7.04	**16.7419**	10	5.01	7.48	**16.5425**	9	3.72	**13.5130**	8	4.01	11.03	**12.7745**
(0, 1)	7	7.94	**10.6136**	7	6.01	8.05	**10.4822**	6	4.13	**8.4243**	5	5.01	9.8	**8.2084**
(0, 1.50)	4	9.06	**5.4827**	3	6.01	10.13	**5.4271**	3	4.82	**4.2280**	3	5.01	12.92	**4.0174**
(0.50, 0.50)	19	5.94	**30.9906**	19	4.01	6.26	**30.8542**	16	3.15	**25.8896**	19	3.01	11.79	**25.1914**
(0.50, 0.75)	13	6.70	**20.3251**	12	5.01	7.03	**20.1340**	11	3.52	**16.5768**	10	4.01	9.16	**15.6954**
(0.50, 1)	9	7.44	**13.3533**	8	5.01	8.06	**13.1842**	7	3.97	**10.6660**	7	4.01	14.01	**10.2151**
(0.50, 1.50)	5	8.61	**6.7608**	4	6.01	9.37	**6.6227**	4	4.53	**5.2821**	3	5.01	12.92	**5.0507**
(0.75, 0.75)	11	7.04	**16.7419**	10	5.01	7.48	**16.5425**	9	3.72	**13.5130**	8	4.01	11.03	**12.7745**
(0.75, 1)	8	7.67	**12.5371**	8	5.01	8.06	**12.3800**	7	3.97	**9.9969**	7	4.01	14.01	**9.5959**
(0.75, 1.50)	5	8.61	**6.9472**	4	6.01	9.37	**6.84236**	4	4.53	**5.4250**	4	5.01	10.81	**5.2153**
(1, 1)	7	7.94	**10.6136**	7	6.01	8.05	**10.4822**	6	4.13	**8.4243**	7	5.01	9.80	**8.2084**
(1, 1.50)	5	8.61	**6.7608**	4	6.01	9.37	**6.6227**	4	4.53	**5.2821**	3	5.01	12.92	**5.0507**
(1.50, 1.50)	4	9.06	**5.4827**	3	6.01	10.13	**5.4271**	3	4.82	**4.2280**	3	5.01	12.92	**4.0174**

From Table2 and as well from Table3, observe the following.

Result 3: ATS_1 of *IWL*-χ^2chart is the least among the related three charts.

Result 4: When $\delta_{1x} = 0$ and $\delta_{1y} > 0$, for high correlation (ρ = 0.7), ATS_1 values of *IWL*-χ^2 chart are significantly smaller as compared to those for (ρ = 0.5).

5. Conclusions

To summarize, *IWL*-χ^2chart detects shifts sooneras compared to the χ^2 chart, *MCCWL*-.χ^2, and *ICC*-.χ^2. For all 14 combinations of $\underline{\delta}_1$, $ATS_{1iwl\text{-}ch} \leq ATS_{1icc\text{-}ch} \leq ATS_{1mccwl\text{-}ch} \leq ATS_{1ch}$. In other words, *IWL*-$\chi^2$ chart performs significantly better as compared to the related three charts. Also, *UCL* of χ^2 chart and *UCL* of *MCCWL*-$\chi^2$is smaller as compared to *UCL* of *IWL*-χ^2chart. As $k_{1iwch} < k_{1mch}$ and $k_{mch} < k_{iwch}$, it indicates the warning region of *IWL*-χ^2chartis wider as compared to that of *MCCWL*-χ^2.

Implementation/operation of the stopping rule for the *IWL*-χ^2 chart is much useful. The proposed chart seems to be used by the industries.

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